The reader is referred to the following source: Shapiro, I. I., *The Prediction of Ballistic Missile Trajectories from Radar Observations* (McGraw-Hill Book Co. Inc., New York, 1958), pp. 93–98.

Comment on "Orbit Decay Characteristics Due to Drag"

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A RECENT paper by Parsons[1] calculates the rotation of line of apsides and the other orbital characteristics purely due to the atmospheric drag for both small and large values of the eccentricity ϵ of the trajectory. However, in his initial setup of the equations of motion, he neglected a dr/dt term which is of the order of ϵ. This can be shown from the Keplerian trajectory, the radius, r:

$$r = \frac{a(1-\epsilon^2)}{1-\epsilon\cos\theta} = \frac{[r^2(d\theta/dt)]^2}{g_0R^2(1-\epsilon\cos\theta)} \tag{1}$$

where θ is the angle of polar coordinates, g_0 is the gravitational acceleration at sea level, R is the radius of the earth, and a is one half of the sum of perigee and apogee radius. So,

$$\frac{dr}{dt} = \frac{a(1-\epsilon^2)\epsilon\sin\theta}{(1-\epsilon\cos\theta)^2}\frac{d\theta}{dt} = \epsilon\sin\theta\left[\frac{g_0R^2}{a(1-\epsilon^2)}\right]^{1/2} \tag{2}$$

Hence, dr/dt is the order of ϵ. Thus, his resulting solutions are in error for terms of order ϵ^2 and higher, and the final conclusions are valid only for orbits of small eccentricity.

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[1] Parsons, W. D., "Orbit decay characteristics due to drag," ARS J. **32**, 1876–1881 (1962).

Author's Reply to Comment by Jain-Ming Wu

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WU'S comment is well taken, as far as it goes. However, as pointed out in the subject paper,[1] the neglected term, $(dr/dt)^2$ [Eqs. (3) and (4)], is multiplied by the air density ρ, which decreases swiftly in an exponential manner as the $(dr/dt)^2$ values increase significantly. Moreover, the slight effect of neglecting the $(dr/dt)^2$ is to reduce the drag impulse so that the magnitude of the derived results may be slightly low.

At the risk of belaboring the point, an extreme numerical example, given below, shows the uselessness of retaining the $(dr/dt)^2$ terms. Wu shows that

$$\left(\frac{dr}{dt}\right)^2 = \frac{\epsilon^2g_0R^2}{a(1-\epsilon^2)}\sin^2\theta \tag{1}$$

In a similar manner

$$\left(r\,\frac{d\theta}{dt}\right)^2 = g_0R^2\,\frac{(1+\epsilon\cos\theta)^2}{a(1-\epsilon^2)} \tag{2}$$

The value of θ, where the total speed is in error by, say, 1% due to the neglect of the $(dr/dt)^2$, is calculated using Eqs. (1) and (2). Recognizing that the last term under the radical is very small compared to unity, expansion gives

$$\left[1+\frac{(dr/dt)^2}{(rd\theta/dt)^2}\right]^{1/2} = 1.01 = 1+\frac{1}{2}\,\frac{\epsilon^2\sin^2\theta}{(1+\epsilon\cos\theta)^2} \tag{3}$$

Since Wu is concerned about the large eccentricities, Eq. (3) is studied, using the extreme case of the escape parabola where $\epsilon = 1$.

$$\frac{\sin^2\theta}{(1+\cos\theta)^2} = \frac{1}{50} = \frac{1-\cos\theta}{1+\cos\theta} \tag{4}$$

This relation shows that the $(dr/dt)^2$ term becomes as important as 1% of the total velocity at $\theta \cong 16°$.

Consider next the altitude change that occurs during that 16° of motion. The equation of the parabola for the extreme case of perigee at the earth's surface is

$$r = 2R_0/(1+\cos\theta) \tag{5}$$

The altitude is determined, using Eqs. (4) and (5) as

$$r - R_0 = R_0\left[\frac{1-\cos\theta}{1+\cos\theta}\right] = \frac{R_0}{50} \cong 69 \text{ naut miles} \tag{6}$$

Since the density decreases by the factor e about every 23 naut miles, ρ is down by the factor $e^{-3} = \frac{1}{20}$.

Therefore, it seems reasonable to suggest that the error in describing the drag, due to neglecting the $(dr/dt)^2$ term, occurs at altitudes high enough to cause a negligible effect on the total drag pulse.

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[1] Parsons, W. D., "Orbit decay characteristics due to drag," ARS J. **32**, 1876–1881 (1962).

Comments on a Hanging Soap Film

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IN a recent note,[1] an apparent paradox has been encountered in the study of a soap film hanging on a horizontal circular frame. If one assumes that the "tensile force per unit length" within the film is constant at every point, the equations for static equilibrium of an element of area are not consistent. It is known, however (see, e.g., Ref. 2), that films, foams, etc., are stable in a gravitational field only if the surface energy (i.e., surface tension) is variable over the surface. The surface energy for a pure substance depends, essentially, on the temperature only, whereas for a liquid mixture it depends strongly on the relative concentrations of the constituents as well.

Films such as that under consideration are observed to be stable only if a liquid mixture, e.g., a soap solution, is used. Hence, when the equilibrium conditions are investigated, variations in the surface tension must be accounted for; the simplest example is a flat vertical film. In the present case, one may take T (in the notation of Ref. 1) to be a function of r only. The equations of horizontal and vertical equilibrium subsequently are found to be

$$\frac{d\phi}{dr} + \frac{\tan\phi}{r}\left(1 - \frac{d\ln T}{d\ln r}\right) = 0$$

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$$\frac{d\phi}{dr} + \frac{\tan\phi}{r}\left(1 + \frac{d\ \ln T}{d\ \ln r}\right) - \frac{w}{2T\cos^2\phi}$$

from which follows the necessary condition

$$dT/dr = w/2\ \sin 2\phi$$

The last relation determines the manner in which T varies if static equilibrium exists. Thus the paradox appears to be resolved; the more difficult problem of determining the shape of the film remains.

References

[1] Gellatly, R. A., "A note on a soap-film paradox," J. Aerospace Sci. **29**, 1487 (1962).

[2] Adam, N. K., *The Physics and Chemistry of Surfaces* (Oxford University Press, London, 1941), 3rd ed., Chap. III.

Comments on "The Adjoint Method and Its Application to Trajectory Optimization"

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A STUDY of the problem of a minimum time transfer between two coplanar circular orbits assuming a constant thrust acceleration has been made, utilizing the Gradient Technique.[1] Optimal steering programs were obtained for values of (thrust acceleration)/(initial gravitational acceleration) from 0.05 to 3.0 and for (radial transfer distance)/(initial orbit rad) values from 0.05 to 1.0. Numerous initial steering programs were used, including those of the form presented by Faulders[2] and by Jurovics and McIntyre.[3] In all cases, the optimal steering programs obtained were of the form presented by Faulders. Thus the solution presented by Jurovics and McIntyre appears to be erroneous.

For the values of distance ratio (0.5) and acceleration ratio (0.166667) considered by Faulders, a (minimum transfer time)/(time per radian in initial orbit) equal to 3.20608 was obtained. This is substantially in agreement with Faulders' value.

For the values of distance ratio (0.1628) and acceleration ratio (1.7343) considered by Jurovics and McIntyre, a (minimum transfer time)/(time per radian in initial orbit) equal to 0.60946 was obtained. This is a substantial improvement over their solution.

The terminal constraint errors for each of the foregoing solutions were less than $3 \times 10.^{-5}$

Solutions have been obtained using an analog computer as well as the IBM 7090.

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References

[1] Bryson, A. E. and Denham, W. F., "A steepest-ascent method for solving optimum programming problems," Raytheon Co. BR-1303 (August 10, 1961).

[2] Faulders, C. R., "Minimum time steering programs for orbital transfer with low thrust rockets," Astronaut. Acta **3**, Fasc. 1 (1961).

[3] Jurovics, S. A. and McIntyre, J. E., "The adjoint method and its application to trajectory optimization," ARS J. **32**, 1354–1358 (1962).

Errata: Damping of a Gravitationally Oriented Two-Body Satellite

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THE following errors occurred in Ref. 1:

In expressions [1–3], replace $t_0/T \leq$ by $t_0/T \geq$.

The sentence preceding inequality [1] should read: "The principal result of the paper is the following upper bound on damping rate (lower bound on damping time):"

In the line preceding inequality [2], "upper bound" should be changed to "lower bound."

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[1] Zajac, E. E., "Damping of a gravitationally oriented two-body satellite," ARS J. **32**, 1871–1875 (1962).